Arhe XVII, 34/2020
UDK 16 Gardner M.
165.412
DOI https://doi.org/10.19090/arhe.2020.34.47-60
Originalni naučni rad
Original Scientific Article

GORAN RUJEVIĆ[1]

University of Novi Sad, Faculty of Philosophy

GARDNER'S PARADOX AND THEORY OF DESCRIPTIONS

Abstract: Martin Gardner's two-children paradox posits two scenarios, in one we know that of two children one is a girl, and in the other we know that of two children the older one is a girl. The chances of the other child being a girl is not the same in these two scenarios, in the first being 1 in 3 while in the second they are 1 in 2. Gardner himself believed that the problem of this paradox lies in the ambiguous way the scenarios are articulated. However, it is possible to show that the original version of the paradox provides sufficient content for a meaningful explanation of these unexpected results. Inspired by comments by Leonard Mlodinow, we attempt to provide a comprehensible explanation for this counterintuitive change with help of Bertrand Russell's theory of descriptions. The difference between the two scenarios then boils down to the difference between indefinite and definite descriptions.
Keywords: definite description, indefinite description, naming, probability, two-child paradox

Few things are as difficult for our brains to intuitively grasp than the various aspects of probability. Even a mundane occurrence such as rain may seem confusing when the weather forecast announces that tomorrow's chance of precipitation is, say, 25%. If we are to spend the majority of the following day out of doors, is it recommended to definitely bring an umbrella for the expected six hours of rainfall, or does this mean that we have 3 in 4 chances of avoiding a downpour altogether,

[1] Author’s e-mail address: goran.rujevic@ff.uns.ac.rs

making the prospect of leaving without an umbrella a reasonable risk to take?

Nowhere is the difficulty of comprehending probability more evident than in games of chance. Entire chains of casinos are built on people's tendency to overestimate the chances of winning and underestimate the likelihood of losing. There is a vast array of psychological traps used to ensnare players with the near-miss effect and sunk cost fallacy. Of course, we would be remiss not to mention the well-known gambler's fallacy, which so often reminds us that a coin doesn't remember: even after ten consecutive heads, when a tails would seem long overdue, the probability is still 50:50.

These issues are not exclusive to gambling, for in any case where there is an uncertain outcome, chances are people will find a way to argue over probabilities. A rather amusing example can be found in *Magic: the Gathering* card game, where some players of mono-coloured decks sometimes include fetchlands in their mana base solely for the purposes of deck thinning. While the fetchlands undoubtedly thin the deck, is the miniscule effect worth the cost of a single life point (to say nothing of the cost on one's wallet)?

However, all of the pitfalls we have mentioned so far are psychological problems of intuitively grasping probability. It should really come as no surprise that the way we "feel" probability should work isn't necessarily the way the world functions. Drawing that fourth land from the library in a row may feel unfair and we may be tempted to cut some lands from the deck, but it might also just be par for the deck's variance. Surely, the best remedy for that is an unbiased and rigorous mathematical analysis of the situation in question, as no person of sound mind would dare argue against cold and definitive numbers.

And yet, challenges with understanding probability are not just false first impressions that are easily sorted out with rigorous methods. Sometimes the very formal mathematical treatment of probability seems questionable or counter-intuitive. These situations are sometimes called "paradoxes", even though they are not paradoxes in the strict sense of the word, but more often than not they are ambiguously worded problems that can have diverging interpretations. Arguably the most interesting of these and perhaps the one which best demonstrates the shortcomings of our probabilistic intuitions is Martin Gardner's "The Two Children Problem", also known as "Boy or Girl Paradox".

This is a two-part problem that can be found in Gardner's *The Second Book of Mathematical Puzzles and Diversions.* The first part goes as follows: "Mr. Smith had two children. At least one of them is a boy. What is the probability that both children are boys?"[2] Assuming the equal likelihood of two outcomes (a boy and a girl), the intuitive answer would be 50:50, or 1 in 2 chances, as the genders of the two children are independent events of equal probability. This, however, is not the correct answer, which becomes evident after we consider the sample space of all possible cases: 1) a boy and a boy; 2) a boy and a girl; 3) a girl and a boy; 4) a girl and a girl. Given that one child is a boy, this eliminates the fourth case and we are left with three possibilities, of which only one (the first one) corresponds to the question posed, thereby making the chances for there being two boys only 1 in 3.

So far, so good, it would seem that this is just another case where our intuitions have led us astray. But the real conundrum appears with the second part of the problem, which goes: "Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls?"[3] All the same assumptions still apply. As a matter of fact, other than the gender in question being reversed (asking about girls instead of boys) and the added information about the older child, this question seems identical to the previous one. It is inconceivable that the age of one child would have any bearing on the gender of the other, thus the answer would surely have to be the same as before, namely, 1 in 3 chances. Yet, it is not. Again, considering the same sample space and positing that the sequence of entities corresponds to the order of their birth, of the four possibilities, numbers one and two are eliminated (as the older child in them is a boy), leaving us with only two cases where the older child is a girl, of which only one contains two girls, thereby making the probability 1 in 2.

---

[2] Martin Gardner, *The Second Scientific American Book of Mathematical Puzzles & Diversions*, The University of Chicago Press, Chicago, 1987, p. 152.
[3] *Ibidem*, p. 153.

The "paradox" here isn't so much that our intuitions deceived us over and over, but rather that the two seemingly identical questions have such diverging answers. We can even go so far as to amend the varying genders in the questions so that this discrepancy becomes even more evident:

P1.1: Of two children, one is a girl. What are the chances that the other is a girl? (1 in 3)

P1.2: Of two children, *the older* one is a girl. What are the chances that the other is a girl? (1 in 2)

How can the addition of the seemingly inconsequential information in italics so drastically affect the answers?

Gardner claimed that the issue is due to "ambiguity arising from a failure to specify the randomizing procedure."[4] In other words, the issue would disappear if the problems were posed in a different manner, perhaps by not talking about a couple of families in a vacuum, but rather by specifying that we are randomly sampling from a pool of families with two children of specified characteristics. However, even this forced frequentist interpretation only helps with quelling the disputes about calculation methods. It still fails to explain how a simple stipulation of relative age causes the change.

A clue towards the answer can be gleaned from the variation of the problem put forth by mathematician Leonard Mlodinow in his book *The Drunkard's Walk*. This version sets out in a familiar way, but deviates from the original soon enough:

P2.1: In a family with two children, if one of the children is a girl, what are the chances that the other one is a girl, too? (1 in 3)

P2.2: In a family with two children, *if we learn that one of the children is a girl with an unusual name (for instance, Florida)*, what are the chances that the other child is a girl, too? (1 in 2)[5]

Mlodinow's take on the problem is markedly different from Gardner's. The second question now doesn't restrict the sample space by

---

[4] *Ibidem*, p. 226.

[5] Slightly adapted from Leonard Mlodinow, *The Drunkard's Walk*, Pantheon Books, New York, 2008.

putting children into ordered pairs. Yet the curious transformation of probability between questions persists. Mlodinow claims that the change from P2.1 to P2.2 demonstrates that the knowledge of a name is *de facto* added information that ought to change how we understand the entire setup. Effectively, Mlodinow implies a Bayesian interpretation of the problem, where the posterior is different from the prior after evidence is introduced. This is parallel to the well-known Monty Hall problem, where opening a door with a goat imparts new information about the system and prompts the contestant to switch doors in hopes of winning an automobile over the other goat.

However, this "new information" explanation still fails to address the issue why an information about the name (or relative age, for that matter) would have any bearing on the probability of gender. One way of dealing with this is to note that any additional information that doesn't provide evidence *against* the case of two girls inherently raises the probability of that case.[6] But this merely takes Gardner's paradox and turns it into Hempel's paradox. Furthermore, there are serious disputes whether Mlodinow actually uses a Bayesian approach, or if his version of the problem is just a different way of restricting the sample space as in Gardner's version.[7]

Perhaps there is a less contentious way to resolve this problem, perhaps we can use the fact that Gardner's and Mlodinow's versions of the problem are different, while still ending up with the same solutions. In both cases the first question refers to a nondescript child, while the second question singles out one child, Gardner by age and Mlodinow by name. Is there a frame of reference where an attribute and a name intertwine? Or, in the words of the infatuated Juliet – what's in a name, anyway?

Mlodinow's inkling that the difference stems from having a named versus unnamed child is is reminiscent of how first-order logic differentiates between individual constants and variables. Perhaps first-

[6] Keith Parramore and Joan Stephens, "Two girls – the value of information" in: *The Mathematical Gazette*, Vol. 98, No. 542, 2014, p. 244.

[7] Stephen Marks and Gary Smith, "The Two-Child Paradox Reborn?" in: *Chance*, Vol. 24, No. 1, 2011, p. 58.

order logic might just help us disentangle this conundrum. If we can find a way to express the various versions of the paradox in the language of first-order logic and then find a suitable model for each version, we just might shed some light on the underlying mechanism that effects the change in probability calculations between the two questions. We need to be careful, however, as not every phrase that constitutes Gardner's paradox should be considered a part of the formal theory; some supply the information needed to create a model for that theory. Therefore, our analysis of each variation must always begin with sorting these elements apart.

Since this undertaking was inspired by Mlodinow's version of the problem, let us begin by taking a look at P2.1 and P2.2. For starters, the common initial assumption that there are two children in question can be considered a declaration of the universe of discourse: we can even give appropriate names to these children, say Addison and Billie. Individual constants associated with them can be a and b, respectively. The only predicate that is used is "is a girl", which can be formalised as G. Even though we assume the possibility that a child can be a girl or a boy, we do not require a separate predicate for "is a boy"; due to the simplistic assumption that a child can be either a girl or a boy, "is a boy" is the same as "is not a girl", so $\neg G$ will suffice. We now posses the basic elements for interpretation according to which we will seek models for the theory we are yet to establish. The phrase that asks about the chances of two girls is merely a stipulation that instruct us to count the number of models that contain no $\neg G$ (no non-girls, that is two girls) against the total number of models.

All that remains from the initial questions are now phrases that talk about which child is a girl. We will now adapt these phrases into what we shall call *core propositions* of the problem, and these will then be translated into first-order logic. These are:

P3.1: There exists at least one girl.
P3.2: Addison is a girl.

The P3.1 proposition is a simple existential statement that does not refer to any entity in particular, and is thus best translated into a formula that uses a variable bound by an existential quantifier:

$$\exists x Gx \tag{F1}$$

Interpreting this proposition in the given universe, the following combinations are consistent with it: $Ga \wedge Gb$, $Ga \wedge \neg Gb$, $\neg Ga \wedge Gb$. The only interpretation that would contradict the proposition F1 is $\neg Ga \wedge \neg Gb$, as there would be no entity left that could satisfy the existential statement. Of the three models, only one of them contains no negations (no non-girls, i.e. boys). Assuming that all possible combinations are equally likely, this gives us the expected probability of two girls as 1 in 3.

The P3.2 proposition refers to a specific entity, Addison, so the formula must use an individual constant as its argument, and no quantifier will be necessary:

$$Ga \tag{F2}$$

The only two combinations that are consistent with this proposition are $Ga \wedge Gb$ and $Ga \wedge \neg Gb$, as any combination that includes $\neg Ga$ directly contradicts it. Of the two models, only one contains no non-girls. Ergo, the expected probability of two girls is in this case 1 in 2.

Let us now turn to Gardner's version of the problem as given in P1.1 and P1.2. Again, we have the declaration of the universe of discourse, and we can again use Addison and Billie here. The "is a girl" predicate *G* makes another appearance, with all the specifics regarding $\neg G$ and boys. However, we also find a new predicate, "is older", which is a contraction of "is older than". This is a binary asymmetric relation that can be represented with *O* such that *Oxy* means "*x* is older than *y*". Stipulations for counting models are the same. We are now left with Gardner's core propositions, which are:

P4.1: There exists at least one girl.

P4.2: There exists at least one entity that is older than the other and is a girl.

The P4.1 proposition is identical to P3.1, so the formula F1 will perfectly apply to it, along with the conclusion of the 1 in 3 probability of both entities being girls.

The issue now arises with proposition P4.2, and there are two ways we can deal with it. We can take it at face value, in which case the corresponding formal expression is:

$$\exists x \exists y (Oxy \wedge Gx) \tag{F3}$$

We can now examine how individual children can slot into this statement. Consistent interpretations include $Oab \wedge Ga \wedge Gb$, $Oab \wedge Ga \wedge \neg Gb$, $Oba \wedge Gb \wedge Ga$ and $Oba \wedge Gb \wedge \neg Ga$, two of which contain no non-girls, giving us the chances of 1 in 2.

The second way we can address the P4.2 proposition is to force an absolute meaning of the words "is older than" so that we can avoid the use of *O* relation. This is feasible, for instance, by tacitly assuming that Addison's and Billie's parents name their children in alphabetical order. With this assumption, the age attribute becomes an inherent characteristic of a child's name, effectively turning that property into an equivalent of a name. The older entity is, by this assumption, Addison, and singling out the older entity is the same as singling out Addison. In this case, the P4.2 proposition is equivalent to the P3.2 version, therefore, the F2 formula would apply, along with the 1 in 2 chances of both entities being girls.

So far, our results seem to be mixed. On the one hand, the use of variables versus the use of individual constants perfectly mimic the results of Mlodinow's version of the problem. On the other hand, Gardner's version is not as clean-cut. When dealing with the P4.2 core proposition, we had to use either a binary predicate or the assumption of alphabetical naming, both of which are elements that do not exist in Mlodinow's version. These novel elements interfere with our conclusions insofar as we cannot be sure if the probabilistic change arises due to the use of individual names versus general descriptions, or due to these confounding novel elements. It would seem that we need to add another layer to our investigation that will account for the relationship between names and individual properties. Fortunately for us, this has long been a point of interest for philosophy of language, and we have ample theories to choose from.

Bertrand Russell's theory of descriptions is arguably most fitting for the circumstances of the problem we are facing. This theory attempts to provide a reliable framework for understanding how various kinds of descriptive expressions of language refer to objects of reality, especially regarding issues such as expressions that do not refer to anything or multiple expressions that refer to one and the same object. The most significant point of interest for us is the fact that this theory includes a comparison of how names and descriptions function. At the very beginning of Russell's seminal text *On Denoting*, we can find a list of three kinds of denoting phrases:

"(1) A phrase may be denoting, and yet not denote anything; *e. g.* 'the present King of France'.
(2) A phrase may denote one definite object; *e. g.* 'the present King of England'.
(3) A phrase may denote ambiguously; *e. g.* 'a man' denotes not many men, but an ambiguous man."[8]

The second kind is what Russell called *definite descriptions*, and he believed that it encompasses not only phrases that refer to a singular entity, but also proper names, which can be seen as shorthands for some more complicated phrases. The third kind of denoting phrases are called *indefinite descriptions*. When analysed in first-order logic, definite and indefinite descriptions show syntactic differences.

If we look back at the core propositions of Gardner's paradox, we can see that they neatly fall into these two categories: P1.1 contains an

[8] Bertrand Russell, "On Denoting" in: *Mind*, Vol. 14, No. 56, 1905, p. 479. It should be noted that Russell wrote this text during the reign of Edward VII, when England indeed had a King.

indefinite description ("one [child]", which is, effectively, the same as "a child"), while P1.2 contains a definite description ("the older child"). This remains stable for all the different versions of these core propositions that we used so far ("one child" and "one entity" are indefinite, while "Florida", "Addison" and "the older entity" are definite descriptions).

Russell posits that indefinite and definite descriptions have different symbolic forms. For instance, the phrase "I met a man" effectively means "there is an *x* such that I met *x* and *x* is human." Therefore, indefinite descriptions have the form $\exists x(Px \wedge Rx)$. On the other hand, the phrase "The father of Charles II was executed" effectively means "There is an *x* such that *x* begat Charles II, and for every *y* if *y* begat Charles then *y* is *x*, and *x* was executed." Therefore, definite descriptions have the form $\exists x(Px \wedge \forall y(Py \rightarrow y=x) \wedge Rx)$. Let us now try to apply these forms to statements of Gardner's paradox.

We begin with all the same assumptions as before. The universe of discourse encompasses Addison and Billie, *a* and *b*, we will use *G* for "is a girl" and add *C* for "is a child". This time, however, instead of using the binary relation *O* for "is older than", we will regard the P1.2 proposition as if it contains a simple description "is the older child". To that end, we will represent it with the unary predicate *S* which carries a simple caveat: in a universe of only two entities, only one may carry the "is older" description, therefore $\forall x \forall y(Sx \rightarrow \neg Sy)$. Per Russell's suggestion, the remaining core propositions are paraphrased as follows:
P5.1: There is an *x* such that *x* is a child and *x* is a girl.
P5.2: There is an *x* such that *x* is the older child, and for every *y* if *y* is the older child then *y* is *x*, and *x* is a girl.

The P5.1 proposition then has the form:

$$\exists x(Cx \wedge Gx) \tag{F4}$$

Since the other child (*y*) can either be a girl or not be a girl, we can conjoin each of these possibilities with the previous proposition, like so:

$$\exists x(Cx \wedge Gx) \wedge \exists y(Cy \wedge Gy) \tag{F5}$$

$$\exists x(Cx \wedge Gx) \wedge \exists y(Cy \wedge \neg Gy) \tag{F6}$$

Since the two variables x and y can either be Addison (a) or Billie (b) in this universe, we need to examine every combination of constants. The formula F5 can be interpreted either as $(Ca \wedge Ga) \wedge (Cb \wedge Gb)$ or as $(Cb \wedge Gb) \wedge (Ca \wedge Ga)$, but these two options are completely equivalent, so formula F5 has only one model. Formula F6 is slightly different, as it can be interpreted either as $(Ca \wedge Ga) \wedge (Cb \wedge \neg Gb)$ or as $(Cb \wedge Gb) \wedge (Ca \wedge \neg Ga)$, which are two different models. Therefore, we have three distinct models, only one of which contains no $\neg G$ (that is, two girls), so the probability of two girls is again confirmed as 1 in 3 chances.

The P5.2 proposition is slightly more complex. It begins with the form:

$$\exists x(Sx \wedge \forall y(Sy \rightarrow y=x) \wedge Gx) \qquad \text{(F7)}$$

If we now wish to expand it with the account of the other child (y), we must remember that predicate "is the older child" cannot be true for both entities, that is $\forall x \forall y(Sx \rightarrow \neg Sy)$. Therefore, the expanded conjunction can only have these forms:

$$\exists x(Sx \wedge \forall y(Sy \rightarrow y=x) \wedge Gx) \wedge \exists y(\neg Sy \wedge \forall x(\neg Sx \rightarrow x=y) \wedge Gy) \qquad \text{(F8)}$$

$$\exists x(Sx \wedge \forall y(Sy \rightarrow y=x) \wedge Gx) \wedge \exists y(\neg Sy \wedge \forall x(\neg Sx \rightarrow x=y) \wedge \neg Gy) \qquad \text{(F9)}$$

Furthermore, since Sy and $\neg Sx$ are necessarily false, both expressions $\forall y(Sy \rightarrow y=x)$ and $\forall x(\neg Sx \rightarrow x=y)$ are necessarily true, so we can eliminate them in order to simplify the formulas into:

$$\exists x(Sx \wedge Gx) \wedge \exists y(\neg Sy \wedge Gy) \qquad \text{(F10)}$$

$$\exists x(Sx \wedge Gx) \wedge \exists y(\neg Sy \wedge \neg Gy) \qquad \text{(F11)}$$

These can now have the following interpretations. Formula F10 can lead either to $(Sa \wedge Ga) \wedge (\neg Sb \wedge Gb)$ or to $(Sb \wedge Gb) \wedge (\neg Sa \wedge Ga)$, while formula F11 can lead either to $(Sa \wedge Ga) \wedge (\neg Sb \wedge \neg Gb)$ or to $(Sb \wedge Gb) \wedge (\neg Sa \wedge \neg Ga)$. Of these four models, only the two that interpret the F10 formula do not contain $\neg G$ (that is, they contain two girls), so the probability of two girls is again confirmed as 1 in 2 chances.

We can now finally pinpoint the origin of the probabilistic difference between the two questions of Gardner's paradox. The question that stipulates that *a child* is of a specific gender contains an indefinite description, but the question that stipulates that *the older child* is of a specific gender contains a definite description. Syntax of the definite description gives four possible interpretations, two of which satisfy the conditions of the question. However, syntax of an indefinite description results in two of the four interpretations being equivalent, and these just so happen to be the two that satisfy the condition of the question, thereby changing the way probability is calculated.

An astute reader may reprimand us for not addressing the role of the $\forall x \forall y(Sx \rightarrow \neg Sy)$ assumption. Is this assumption not a novel element in the analysis and thus another possible culprit of the change of probability? To that we would answer that this is indeed true, however, this assumption does nothing else but establish the uniqueness of the "is the older child" description in a two-entity universe of discourse. In other words, the definite description found in P5.2 is facilitated by this very assumption. Therefore, pointing at this assumption as a possible culprit is just a roundabout way of pointing again at the distinction between indefinite and definite descriptions.

Even though Russell's theory of descriptions was created with more profound philosophical issues in mind, it is still pleasing to see how it can be used to shed light on the unusual case of Gardner's paradox. If anything, we can now say that we have conclusive proof that this issue of probability is not a paradox, but a curious case that challenges our superficial intuitions. The possibility of establishing a clear separation of the two questions on the account of indefinite and definite descriptions used in them effectively shows that Gardner's own remark on the ambiguous nature of the problem was in a sense dispensable, as the problem can be made transparent as is. Of course, it is almost trivially true that every problem can be made more accessible if articulated in a more exhaustive manner.

Ultimately, there are two additional benefits to uncovering the mechanism underlying the Gardner's paradox. Firstly, it offers a clear rebuttal to any claim that Gardner's paradox is fallacious or that Gardner made a mistake while reasoning about it.[9] Secondly, providing a reason for an otherwise counterintuitive situation is fundamentally satisfying. It is why paradoxes, puzzles and enigmas are so interesting in the first place. When facing a situation that defies our expectations, instead of passively hoping that we will simply get used to it as some kind of "new normal", it is always far better to put in the effort to see what the case really is in order to better manage our expectations in the future. The former approach is complacency, the latter is curiosity.

REFERENCES

Gardner, Martin, *The Second Scientific American Book of Mathematical Puzzles & Diversions*, The University of Chicago Press, Chicago, 1987.

Khovanova, Tanya, "Martin Gardner's Mistake" in: *The College Mathematics Journal*, Vol. 43, No. 1, 2012.

Marks, Stephen and Smith, Gary, "The Two-Child Paradox Reborn?" in: *Chance*, Vol. 24, No. 1, 2011.

Mlodinow, Leonard, *The Drunkard's Walk*, Pantheon Books, New York, 2008.

Parramore, Keith and Stephens, Joan, "Two girls - the value of information" in: *The Mathematical Gazette*,Vol. 98, No. 542, 2014.

Russell, Bertrand, "On Denoting" in: *Mind*, Vol. 14, No. 56, 1905.

[9] For instance, Tanya Khovanova, "Martin Gardner's Mistake" in: *The College Mathematics Journal*, Vol. 43, No. 1, 2012.

GORAN RUJEVIĆ

Univerzitet u Novom Sadu, Filozofski fakultet

GARDNEROV PARADOKS I TEORIJA DESKRIPCIJA

Sažetak: Paradoks dvoje dece Martina Gardnera postavlja dva scenarija, jedan u kom znamo da je jedno od dvoje dece devojčica i drugi u kom znamo da je starije od dvoje dece devojčica. Šanse da je i drugo dete devojčica nisu iste u ova dva scenarija, u prvom iznose 1 od 3, a u drugom 1 od 2. Sam Gardner je smatrao da je problem ovog paradoksa u nejasnoj artikulaciji same postavke, međutim, moguće je pokazati da i prvobitna verzija paradoksa pruža dovoljno sadržaja za smisleno objašnjenje porekla ovih neočekivanih rezultata. Inspirisani zapažanjima Leonarda Mlodinova [*Leonard Mlodinow*], pokušavamo da pronađemo suvislo objašnjenje ove kontraintuitivne promene pomoću teorije deskripcija Bertranda Rasela. U tom pogledu, razlika između ova dva scenarija se svodi na razliku između neodređenih i određenih deskripcija.

Ključne reči: imenovanje, neodređena deskripcija, određena deskripcija, paradoks dvoje dece, verovatnoća

Primljeno: 18.8.2020.
Prihvaćeno: 11.11.2020.