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VESSELIN PETROV
Department "Ontology and Epistemology"
in the Institute for Philosophical Research
at the Bulgarian Academy of Sciences, Sofia

THE CONCEPTION OF BERNARD BOLZANO ABOUT THE CONTINUUM AND THE ACHIEVEMENTS OF CONTEMPORARY MATHEMATICS

Apstract: The present paper is devoted to the contribution of the eminent philosopher and mathematician Bernard Bolzano to the problem of continuum. The paper consists of two parts. In the first part it is compared the conception of Bolzano about the continuum with that of Aristotle. Bolzano's view includes two essential moments: first, that the continuum is composed of noncontinual elements (points), and second, that two points which are at a certain distance one to other cannot compose a continuum. Aristotle has two definitions about continuity: in the first one he in fact develops the idea of continuity as a physical connectedness, and in the second one he emphasizes the infinite divisibility of the continuum.

The second part of the paper gives an evaluation of Bolzano's contribution to the problem of continuum in comparison with the achievements of the contemporary mathematics on the problem. There is a tendency to consider continuity in two senses: as an infinite divisibility and as a connectedness. The contemporary topology (as a branch of mathematics) develops the concept of continuity in the second sense. Bolzano's conception bears in some respects the resemblance to the contemporary definitions in topology of the concepts of closeness, neighborhood, isolated point, etc. His definition about the continuum can be considered as a remote precursor of the contemporary topological definition of continuity as connectedness.

Key words: Bolzano, continuum, Aristotle, contemporary mathematics.

In the present paper we shall fix our attention on the problem of continuum, the contribution of Bernard Bolzano (1781-1848) to which development is not enough discussed in the philosophical literature. Nowadays the problem of continuum is not only philosophical; it has become a subject of discussion in the contemporary mathematics and physics. However many discoveries made at the end of the XIX-the and the beginning of the XX-the centuries in the fields of mathematics repeat to a certain extent what is done by Bolzano or they are a continuation of his initiatives. At the same time his philosophical gropings in the fields of metaphysics, logic and epistemology have points of contact with or, on the opposite, they dispute with the preceding philosophers who have also worked on the same problem. That is why it is not strange that namely Bolzano who combines in himself the talents of a mathematician, a philosopher and a logician has managed to make an essential contribution to the development of the problem of continuum for which different knowledge and abilities in the fields of philosophy, mathematics and other exact sciences are required.

BOLZANO AND ARISTOTLE

The mature conception of Bolzano about the continuum is best presented in his work "Paradoxes of the infinite" which is written at the end of his life and is published after his death.[1] In our exposition we shall follow that work of Bolzano.

What is a continuum according to Bolzano? At the very beginning of § (section) 38 of "Paradoxes of the infinite" which is fully devoted to the problem of the continuum it becomes clear that he uses the concepts "continuum" and "continual extension" as synonyms. He gives the definition in the following sense: If we try, says Bolzano, to make clear the concept which we name continual extension or continuum, we shall be forced to admit that a continuum exist there and only there where a collection of simple objects is available (points in the time or in the space, or substances) which are arranged in such a way that each of them has at least one neighboring object at every distance however small it may be. If it is not so, if for example in a given collection of points of the space there exists even only one point which is surrounded by neighboring points not so densely that a neighboring point to be available at every enough small distance, then we shall say that this point is an isolated point and that because of this reason the collection is not a perfect continuum.

We see that two main points can be emphasize in Bolzano's view: first, he speaks of a collection of "simple objects" (points or substances) and, second, not a single point is isolated. It is clear from that definition that Bolzano considers the continuum as it is made up of points. We shall come back to that detail later.

Immediately after giving his definition Bolzano enters into a discussion with his eventual opponents. He brings forward as an eventual objection the requirement that "every point to be in a direct contact with another point" and after that he points out the impossibility such a requirement to be satisfied, because it includes a contradiction: What means two points to be in contact? It means that the end of the one point should coincide with the end of the other point. But the points are simple parts of the space and they have no ends, no parts. That's why the requirement a part of the one point to coincide with a part of the other point would mean the points themselves to coincide. But if the one point has something separate from the other, then both points should be perfectly separate one from the other and in such a case there would be at least one point (and consequently infinitely many points) between them and our two points would not be in contact. It is clear from this objection that Bolzano latently debates here with the conception of Aristotle. In such a way we reach according to the logic of Bolzano himself to a comparison of his views with these of Aristotle

[1] Bolzano, B. *Paradoxien des Unendlichen*. Leipzig, 1851. There is also a russian translation: Больцано Б. Парадоксы бесконечного. – Одесса: 1911.

Let us remember what is Aristotle's view about the continuity. Aristotle speaks about continuity; the concept "continuum" have not taken shape at that time yet. So in connection with the conception of Aristotle we shall use the concepts continuum and continuity as synonyms. First of all Aristotle differs continuity as a particular kind of connection from the other kinds - succession and contiguousness. A thing that is in succession and in contact is contiguous. And "a thing is continuous when the extremities of each at which they are in contact become one and the same and are (as the name implies) contained in each other" ("*Physics*", 226 b - 227 a; see also "*Metaphysics*", 1069 a). If things are in contact, but each has its extremities so that two boundaries in contact do not become one, then things are only contiguous; if things in contact have a common boundary, then they become one and here we have continuity. In this definition Aristotle in fact develops the concept of continuity as a physical connectedness, i.e. he as if is guided here first of all by the visual physical perceptions and notions. We must pay attention to the fact that Aristotle's physics allows as it is two types of connection in the physical world without violation of the principle of continuity: 1) continuity in proper sense when two bodies have a common boundary, and 2) a contact when the boundary between two bodies, even if it is not common, but there is nothing in the "gap" between them, i. e. in fact there is no "gap" between them. Proceeding from his view about continuity as a *physical* continuity, Aristotle says that it is impossible something continuous to consist of indivisible parts, e. g. the line of points, if the line is continues and the point is indivisible ("*Physics*", 231a, 24-26). As we saw earlier, Bolzano hints in § 38 of his "*Paradoxes of the infinite*" exactly at that view of Aristotle and adds that it is not possible to touch this, nor to perceive it by the sight, but it is cognized by the mind. In other words, Bolzano is clearly aware of the reasons for the difficulty in Aristotle, namely, the Aristotle's view about the continuum as a physical concept.

Before going on further with the analysis of Bolzano's conception let us recall that Aristotle has also another definition about continuity. Aristotle says that continuous is "that which is divisible into divisibles that are infinitely divisible" ("*Physics*", 232b, 25). If we consider this definition again as a physical one, it does not differ in its essence from the previous definition: it is clear that the continuous excludes indivisible parts, whatever they could be, and all the more it cannot be composed of indivisibles. The continuous has parts each one of which, in its turn, consists again of parts, and the indivisible does not at all consist of parts. That which does not consist of parts, it is clear, cannot be in contact with something which also does not consist of parts, because the concept itself of contact already contains in itself the condition of being divisible into parts: only that is in contact which is divisible, because the extremities only of the divisible can be contained in each other. The indivisible has no extremities, that's why according to Aristotle two indivisibles cannot be in contact.

If we consider points physically (as material points or something like the atoms of Boskovich), then again it is completely understandable the position of Aristotle that if we consider the line as a continuum, as one continuous wholeness, then every point in it is only a boundary of the parts into which it can be divided, and that a continuum as the line does not consist of points that are only common virtual boundaries of two parts of the line and they are, in their turn, also lines. On the other hand however, the Aristotle's continuum is not properly speaking ***physis***, it is only a modality of ***physis***, a concrete being like an animal, a stone, etc. is a continuum and at the same time it has a geometrical structure. In such a way Aristotle as if at least hints at the possibility for the concept of continuum to be understand also as a mathematical concept that differs from the physical concept of continuum. Namely the second Aristotle's definition of continuum (continuous is "that which is divisible into divisibles that are infinitely divisible") is in favour of that hypothesis. Of course, it would be a baseless extrapolation to think that Aristotle has also a concept of continuum as a concept of some general structure. But nevertheless the possibility of such understanding is contained in that second definition of the continuum although it had remained undeveloped and unrealized It is done only in the contemporary mathematics in which it is developed, as we shall see in the second part of this paper, also the conception of continuity as connectedness, and not as a physical connectedness, but in a pure abstract mathematical sense.

It is the right place here to point out a very important difference between the views of Aristotle and Bolzano. We are speaking here of the difference between their views about the infinity that, in turn, has an essential effect also on their views about the continuum. Aristotle is against the actual infinity and he is the founder of the conception of the potential infinity, while Bolzano is, on the opposite, an ardent adherent of the idea of the actual infinity.

Aristotle had grasped the narrowness of his conception about the continuum, but he had tried to find a way out not in the atomism as an opposition to the continuism, but in the idea of the potential infinity. According to the description that he gives in "*Physics*" 3.5-7 the infinity exists only in a narrow sense: the elements of an aggregate are never more than finite in number, though their number can be increased inimitably. The number of the actually realized divisions of an aggregate (e. g. of a line) cannot be infinite. At the same time Aristotle aware that he should assume something more than that and he makes his concession in terms of the potentially existed divisions, supposing that their number is more than finite, because it is compared with the number of the actually existed divisions. In "Physics", 204a, 20-26, he even feels a need to exclude all aggregates that are larger than finite, because he is apprehensive that all such aggregates would contain subaggregates that are infinite in the same sense, and he considers that it is impossilble.

Such is the well-known conception of Aristotle about the infinity, Bolzano, on the opposite, is an ardent adherent of the idea of the actual infinity. It is no accident that he quotes the eloquent thought of G. W. Leibniz as a motto of his basic work "*Paradoxes of the infinite*": "I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author."[2] It is well known that namely the doctrine of Leibniz had a greatest influence on the molding of Bolzano's philosophical views. And Leibniz is an ardent supporter of the actual infinity.

As to Bolzano, no one of the existed definitions of the infinity given before him by other thinkers satisfies him. In his "*Paradoxes of the infinite*" (§§ 2-12) he

[2] Bolzano, B. Op. cit. Cf. Leibniz in a letter to Foushe from 1693 - in: Лейбниц Г.В. Сочинения. Т. 3. – М.: Мысль, 1984, с. 294.

criticizes them and he suggests his own definition of the infinite: "The simplest definition is as follows: an infinite large is that which is greater than any *given* magnitude." (§ 12) It is necessary to define here more exactly what should be understood by the word "given". Bolzano explains a little later that "we understand by this word everything which can be given to us, i. e. everything that can become an object of our experience" (§ 12). He points out in the next section that the nearest question is the question if this concept (the concept of an infinite large) has objectivity, i. e. if there are objects to which it can be applied and his answer is categorically positive. The concepts of an infinite, a set, a number and so on express according to Bolzano the inner properties of the objects and they do not depend on our cognitive faculty. He repeatedly points out the objectivity of these concepts: the set is not created by our mind, it exists in reality. Bolzano emphasizes that the impossibility to think an infinite set of objects together does not mean the impossibility of its objective existence.

It is interesting to consider in what way the accepting only of the potential infinity and the rejecting of the actual infinity by Aristotle, on the one hand, and the fully admission of the actual infinity by Bolzano, on the other hand, have been influenced their conceptions about the continuum. In Aristotle, on the one hand, it is required the elements of a continuous body to merge gradually, and on the other hand, it is required they at the same time to follow one after another. By his physical conception about continuity Aristotle introduces a new type of ordering that combines the idea of a succession and that of contiguousness. He links the idea of a succession with the thesis about only the potential existence of the parts of the continuum. The intuitive basis of the conception of Aristotle about the continuity is the notion of a movement on a trajectory without "gaps" and we should emphasize that namely this image is associated with the idea of a succession, and hence with the idea of the potential infinity. Namely because of the acceptance only of the latter and the rejecting of the actual infinity Aristotle thinks that a continuum as the line does not consists of points which are only potential boundaries of the potential parts into which the line can be divided.

The intention of Bolzano is rather a mathematical one and not a physical one and because of that he proceeds from the concept of function as a formal model of the movement. The concept of function, in turn, is very well developed by Bolzano and in its basis it is the concept of simple correspondence between two sets of arbitrary elements. We must say immediately however that such a correspondence between two arbitrary sets presupposes hiddenly the idea of the actual infinity. Namely the acceptance of the latter makes possible the idea of Bolzano about the building of the continuum as a set of points, a revolutionary idea that has given a powerful impetus to the development of mathematics from that moment on. This idea had turn out to be unattainable for Aristotle not only because the concept of a set (and of an element) had been undistinguished by him from that of an aggregate, but first of all because of the rejection of the actual infinity by Aristotle.

As we pointed out earlier when we expressed the definition of Bolzano about the continuum, his view included two essential moments: first, that the continuum is composed of noncontinual elements (points), and secondly, that two points which are at a certain distance one to other cannot compose a continuum. Bolzano says in connection with the first that it is natural that the whole has properties which its parts do not possess and that in the given case of the continuum the latter is composed of points which are elements of the continuum.

Here the main reason for the continuum to has properties which its elements - the points - do not possess is the fact that according to Bolzano the continuum is an actual infinity of its elements. As to the second moment, Bolzano notices that we must assume an infinity of intermediate points between any two different points, if we want that there would not be a contradiction in the statement that the continuum is composed of points. But again there can be no denying of that supposition, only if we proceed from the idea of the actual infinity, as Bolzano does. In other words, we see that the idea of the actual infinity really plays a decisive role in the conception of Bolzano about the continuum. Without this idea he would not be able to defend his view.

BOLZANO AND THE CONTEMPORARY MATHEMATICS

Let us give now a short evaluation of Bolzano's contribution to the problem of continuum, comparing it with the achievements of the contemporary mathematics on that problem.

After Bolzano has pointed out in § 19 of his *"Paradoxes of the infinite"* that we should not to consider as equal to each other all infinite ensembles with respect to their multitudeness, but some of them are greater, others are less, he says in the next § 20 that "two infinite ensembles can be in such a relation to each other, that on the one hand, it is possible to connect in a pair every element of the one ensemble with some element of the other in such a way that in both ensembles will not remain any element which is not connected in a pair and any one element will not be a member of two or several pairs. On the other hand, it is possible however one of these ensembles to include in itself the other just as a part." The impression is created here that Bolzano speaks about "ensembles". We must take into account that in the very beginning of *"Paradoxes of the infinite"* he makes a distinction between several concepts: a collection of some elements is for him the same as a whole which consists of some parts (§ 3); in § 4 he points out that a collection in which the position of its parts does not matter is an ensemble; and such an ensemble all parts of which are considered as units of a given genus *A,* i. e. as objects contained in the concept of *A,* he calls a set of objects *A.* Bozano defines further sums of collections, magnitude, series, etc. (§§ 5-7 and the next ones) - something which is of no interest to us now. In other words, we see that the contemporary concept of set (introduced later for the first time by Cantor) is not yet "crystallized" with Bolzano. Nevertheless the tendency is clearly seen to find out something in common amongst the infinite sets (ensembles according to his terminology) which would enable to group them as a hierarchy of equivalent set, when it turns out that an infinite set can be equivalent with his proper subset.

When we use our own judgment about the question what had enabled Bolzano to make an important contribution to mathematics and to come close to the creating of the theory of sets, the fact should be stressed that Bolzano is clearly aware of the legitimacy and the fundamental necessity of the concept of the actual infinity. It had enabled him to introduce the concept of the equivalence (which is analogous to the contemporary concept of the equal cardinality of sets) showing the illusory character of the seeming contradictions which some mathematicians and philosophers before him had linked with the infinity. Georg Cantor, beginning to develop his theory of sets, proceeds namely from Bolzano's ideas.[3]

The achievements of the contemporary mathematics show that different kinds of ordering are considered in the theory of sets. A given set that contains the same elements can be ordered in different ways and its cardinality will not change because of that. When we consider the mathematical continuity, then an important role plays namely the ordering of a set. Russell used to say that the continuity of a set is a property of the ordering of his elements: "But continuity, which we are now to consider, is essentially a property of an order: it does not belong to a set of terms in themselves, but only to a set in a certain order. A set of terms which can be arranged in one order can always also be arranged in other orders, and a set of terms which can be arranged in a continuous order can always be arranged in orders which are not continuous. Thus the essence of continuity must not be sought in the nature of the set of terms, but in the nature of their arrangement in a series."[4]

Because of the importance of this distinction, let us recall the exact definition: An order (a partial order) is a transitive relation R. A set A is lineary ordered, if a relation of order R is given on it which has the property of connectedness, i. e. for every $x, y \in A$ the condition $(xRy) \vee (yRx)$ is fulfilled and if from (xRy) and (yRx) it follows that $x = y$. A relation R is a well order *of A,* if and only if it is a linear order of A and every nonempty subset has a smallest element. A cut $<X, Y>$ of a lineary ordered set A is defined by a pair of sets Aand Y, such that $(x \in X) \wedge (y \in Y) \rightarrow (x \leq y)$. Moreover the set $X \cap Y$ has no more than one element; and if it has no elements at all (i. e. if it is empty), then we say that the cut defines a "gap" in the set A. A set A is called continuous, if any one its proper cut (i. e. a cut for which $X \neq \varnothing \neq Y$) does not define a "gap".[5] The continuity is linked with the uncountable sets, because namely such sets can satisfy the condition of the absence of "gaps". Hence, the connection of the continuity with the continuum: the set of all real numbers in the interval [0,1] has the cardinality of the continuum and at the same time it is continuous, because there are no "gaps" in it. But not every set which has the cardinality of the continuum is a continuous one: for example, the set only of irrational numbers in the interval [0,1] has also the cardinality of the contunuum, but it is not a continuous set in the given sense, because it has "gaps": there is a rational number between any two irrational numbers and this rational number does not belong to the set and it defines a "gap" in the corresponding cut of irrational numbers.

We must point out that all that is so on the assumption that the continuum-hypothesis of Cantor is true. There are different acsiomatic systems of the noncontradicted part of the Cantor's theory of sets. One of them is the so called Zermello-Fraenkel system in which an important role plays the statement that every

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[3] G. Cantor. Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Leipzig, 1883. See Dauben, J. W. Georg Cantor. His Mathematics and the Philosophy of the Infinite. Harvard Univ. Press, 1979, p. 110, 123-124.

[4] Russell, B. Our knowledge of the external world. L. and N. Y., Routledge, 1993, p. 138.

[5] See for example the classical book: Kuratowski, K, Mostowski, A. Set theory. Amsterdam-Warszawa, 1967.

set can be well-ordered. Bearing however in mind the independence of the continuum-hypothesis (Kohen, 1963), it is possible to create a non-Zermello mathematics in which sets with the cardinality of the continuum are possible, but in which such sets cannot be well-ordered. There is no "precipice" between the finite and the enumerable infinite as well as between the enumerable infinite and the continuum in such a mathematics, i. e. it is possible to exist even infinitely many cardinalities between the enumerable infinite and the continuum, and different kinds of "concentrations" of the continuity are also possible, beginning from the absolute discreteness to the continuum. In the Cantor-Zermello's theory of sets however the notion of well-ordering plays an extremely great role and, as we pointed earlier, the continuity is so to say a "spetial case" of the order.

The notion according to which a set is continuous, if any one its proper cut does not give a "gap", had been developed by Cantor to the conception of the continuum as a perfect and a connected set of points. A set of points is a perfect set, if the process of obtaining of its derivative set does not change it (the derivative set is the set of its points of accumulation). We call the set *Ta* connected set, if for any two points $t, t' \in$ *Tand* for any $\varepsilon>0$, however small it could be, there exists a finite number of points $t_1, t_2, \ldots t_n \in T$, such that the distances $tt_1, t_1t_2, \ldots t_nt'$ are all smaller than ε.

The specified concept of connectedness is restricted by the fact that it presupposes the presence of a metrix **(**a generalization of the usual notion of a distance). Topology overcomes this restriction giving a more general notion of the connectedness and also a strict definition of the intuitively clear notions about closeness (do not mix with closure), neighbourhood, etc. For example, the concept of "balls" is introduced to define the concept of closeness in the following way: $D_r(x_0) = \{x \in X: \rho(x, x_0) < r\}, r > 0,$ with a centre in the point *XQ* and a radius *r.* Then we say that the point *x* is ε-closed to the point x_0, if $x \in D_\varepsilon(x_0)$. Here *X* is a set (a metric space) and ρ is a metric in it. This concept of closeness enables to formulate exactly the intuitively clear concept of the neighbourhood of a point. Namely, the part Ω of a metric space is a neighbourhood of its point x_0, if any point which is enough closed to xo belongs to Ω. If however a metrix is not introduced in the set *X,* then it is possible to work directly whith topological spaces.

Let in the set X with arbitrary nature of its elements a population $\tau = \{U\}$ of subsets is chosen with the following properties: 1) $\varnothing, X \in \tau$, 2) the union of any population of sets of r belongs to τ, 3) the intersection of any finite number of sets of τ belongs to τ. Such a population of subsets τ is called a topology in *X* and the set *X* is called a topological space and is marked by (X, τ). We call the elements of τ open sets. For the sake of brevity we shall mark the topological space only by *X.* The following definitions are useful: Let *X* is a topological space. A set *U* of the topological space *X* is a neighborhood of the point *x* of *X,* if and only if *U* contains an open set which contains *x.* The point *x* is a point of accumulation of a subset *A* of the topological space *X,* if and only if every neighborhood of *x* contains points of *A* which are different from *x.* The set of all points of accumulation of a given set *A* is called the derivative set of *A* and is marked by *A'.* A subset $A \subset X$ is closed, if and only if it contains all its points of accumulation. The closure $\overline{A}$ of a set $A \subset X$ is the intersection of all closed sets which contain *A.* Now it follows an important for us definition: the point $x \in A$ is an isolated point, if there exists a neighborhood $\Omega(x)$ of the point x which does not contain any points of the set A which are different from x. The set A is a discrete one, if every its point is an isolated point. On the other hand, two subsets A and B of a topological space X are separated in X, if and only if $\bar{A} \cap B = A \cap \bar{B} = \varnothing$. Now we reach to the exact definition of the intuitively clear concept of connectedness: the spaced is *non-connected,* if it can be presented as an union of two nonempty separated one from the other sets. On the opposite, a topological space X is connected, if and only if X cannot be presented as an union of two nonempty separated one from the other sets. For example, every line segment $[a, b]$ is a connected set. However the set of all irrational numbers is not connected. Here the link between the concepts of connectedness and continuity begins to be seen. It can be proved that the graph of a continuous map of a connected space is also a connected set.

If we look at the beginning of this paper to the both views of Aristotle about the continuity, we shall see that a tendency has been presented yet in the ancient philosophy to consider continuity in two senses: as an infinite divisibility and as a connectedness. The contemporary topology, as we have seen from the previous exposition, has developed and has given an exact definition of continuity in the second sense. On the other hand, as we have seen, it is closely connected with the problem of the actual infinity which, in turn, is connected with the problem of the continuum. These two problems have been developed mostly in the set theory, and also in the mathematical analysis, the topology, etc.

What does all this have to do with Bolzano? The exposed quite detailed consideration of the achievements of the contemporary mathematics which are connected with the concepts of continuity and continuum enables us to evaluate the contribution of Bolzano not only by comparing it with the conceptions of the previous philosophers and by seeing what he has done above them, but also by comparing it with the contemporary state of the problems - something which enables us to estimate the perspicacity of Bolzano or, on the opposite, his faults. For example, the contemporary set theory enables us to clarify the fact (for which discovery there were no enough means neither in the Antiquity, nor later) that not only a finite set of points in not enough to form an extention (a continuum) of it, but also not every infinite set of points is enough for it. Bolzano seems to realize it, when he points out the necessity of an adequate definition of an infininte set of points for the formation of a continuum. Let us recall what he says about the continuum in § 38 of his *"Paradoxes of the infinite",* when he discusses the objection of mathematicians against his definition of the continuum who say that the extentive cannot be made of any (even of greatest) concentration of points and that it cannot be decomposed into simple points: "a finite set of points will never compose an extension and an infinite set of points will certainly compose it, but only when the many times mentioned condition is fulfilled, namely, that for every point have to exist some neighboring points at any enough small distance". If we compare his given earlier definition of the continuum which is exposed two pages earlier in his work, we shall see that it bears in some respects the resemblance to the contemporary definitions in topology of the concepts of a closeness, a neighborhood, an isolated point, etc. Of course, these concepts in the formulation of Bolzano have not yet attained its distinctness and exactness that is characteristic of their contemporary formulation. The definition of Bolzano can in some sense be considered as a remote precursor of the contemporary topological definition of the continuity as a connectedness. But regardless of his clear support of the actual infinity he is not successful in constructing of the hierarchy of the infinities, which is realized later by Cantor, mainly because of the unclarified yet concept of a set by him. That's why he does not realize the continuum as possessing a greater cardinality than the enumeralbe one, but he as if is interested much more in the type of ordering of the ensemble that he calls a continuum. In this he as if also anticipates the contemporary considerations in mathematics according to which continuity is a type of ordering. In § 41 of his *"Paradoxes of the infinite",* immediately after the sections in which he develops his conception about the continuum, the space and the time, Bolzano gives examples illustrating his definition, some of which are very interesting. But for the lack of space, we shall not consider them.

There are also inaccuracies in Bolzano's work due to the insufficient "crystallization" of his concepts. We still can find in Bolzano neither the concept of a metric space, nor the concept of a topological space, much less of their differentiation. But what is important in the definition of Bolzano about the continuum is something else: there is a guesswork that the continuity would break, if the connectedness is broken The examples of Bolzano illustrate namely that. As we saw earlier, for the defining of the continuity in the contemporary set theory is accepted just a generalization of the Dedekind cuts which are used by Dedekind himself to define the real numbers. Of course, at the time of Bolzano it is still early for the Dedekind cuts, still more for a generalization of such a construction. The mathematics needed still half a century (for some of the contemporary results even more) to differentiate an ordering from a measure, topology from metrics and to make clear the contemporary concept of the connectedness. However Bolzano has put the first "traverses" initiating the beginning of this long and exciting way.

Veselin Petrov
Akademija nauka, Sofija

BOLCANOVA ZAMISAO KONTINUUMA I DOSTIGNUĆA SAVREMENE MATEMATIKE

Sažetak

Ovaj članak je posvećen poznatom filozofu i matematičaru Bernardu Bolcanu i njegovom doprinosu problemu kontinuuma. Članak se sastoji iz dva dela. U prvom delu je Bolcanov pojam kontinuuma upoređen sa Aristotelovim. Bolcanovo stanovište uključuje dva ključna momenta: prvi, da je kontinuum sačinjen od nekontinualnih momenata (tačaka) i drugi, da dve tačke koje su na izvesnoj distanci jedna od druge ne mogu sačiniti kontinuum. Aristotel ima dve definicije kontinuiteta: u prvoj on zapravo razvija ideju kontinuiteta kao fizičke povezanosti, dok u drugoj naglašava beskonačnu deljivost kontinuuma.

Drugi deo ovog rada nudi evaluaciju Bolcanovog doprinosa problemu kontinuuma u poređenju sa postignućima savremene matematike. Postoji tendencija da se kontinuitet posmatra u dva smisla: kao beskonačna deljivost i kao povezanost. Savremena topologija (kao oblast matematike) razvija pojam kontinuiteta u drugom smislu. Bolcanov pojam u izvesnom smislu podseća na savremenu topološku definiciju pojma zatvorenosti, približnosti, izolovane tačke itd. Njegova definicija kontinuuma može biti posmatrana kao preteča savremene topološke definicije kontinuiteta kao povezanosti.